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Applied Mathematics Letters 20 (2007) 741–745
Applied Mathematics Letters
www.elsevier.com/locate/aml

Global attractivity for a family of nonlinear difference equations[☆]

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Received 16 May 2006; received in revised form 3 August 2006; accepted 11 August 2006

Abstract

In this note, we consider the following nonlinear difference equation:

$$x_{n+1} = \frac{f(x_{n-r_1}, \ldots, x_{n-r_k})g(x_{n-m_1}, \ldots, x_{n-m_l}) + h(x_{n-p_1}, \ldots, x_{n-p_s}) + 1}{f(x_{n-r_1}, \ldots, x_{n-r_k}) + g(x_{n-m_1}, \ldots, x_{n-m_l}) + h(x_{n-p_1}, \ldots, x_{n-p_s})}, \quad n = 0, 1, \ldots,$$

where $f \in C((0, +\infty)^k, (0, +\infty))$, $g \in C((0, +\infty)^l, (0, +\infty))$ and $h \in C((0, +\infty)^s, [0, +\infty))$ with $k, l, s \in \{1, 2, \ldots\}$, $0 \le r_1 < \cdots < r_k$, $0 \le m_1 < \cdots < m_l$ and $0 \le p_1 < \cdots < p_s$, and the initial values are positive. We give sufficient conditions under which the unique equilibrium $\overline{x} = 1$ of this equation is globally asymptotically stable, which extends and includes corresponding results obtained in the cited references.
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Keywords: Difference equation; Global asymptotic stability; Locally stable; Equilibrium; Positive solution

1. Introduction

The study of properties of nonlinear difference equations has been an area of intense interest in recent years (for example, see [1–3]). In [4], Ladas suggested investigating the global asymptotic stability of the following nonlinear difference equation:

$$x_{n+1} = \frac{x_n + x_{n-1}x_{n-2}}{x_n x_{n-1} + x_{n-2}}, \quad n = 0, 1, \ldots, \tag{E1}$$

where the initial values $x_{-2}, x_{-1}, x_0 \in R_+ \equiv (0, +\infty)$.

In [5], Nesemann utilized the strong negative feedback property to study the following difference equation:

$$x_{n+1} = \frac{x_{n-1} + x_n x_{n-2}}{x_n x_{n-1} + x_{n-2}}, \quad n = 0, 1, \ldots, \tag{E2}$$

where the initial values $x_{-2}, x_{-1}, x_0 \in R_+$.

☆ Project supported by NSFC (10461001, 10361001) and NFSGX (0640205).

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doi:10.1016/j.aml.2006.08.024

In [6,7], Li studied the global asymptotic stability of the following two nonlinear difference equations:

$$x_{n+1}=\frac{x_{n-1}x_{n-2}x_{n-3}+x_{n-1}+x_{n-2}+x_{n-3}+a}{x_{n-1}x_{n-2}+x_{n-1}x_{n-3}+x_{n-2}x_{n-3}+1+a},\quad n=0,1,\ldots,\tag{E3}$$

and

$$x_{n+1}=\frac{x_nx_{n-1}x_{n-3}+x_n+x_{n-1}+x_{n-3}+a}{x_nx_{n-1}+x_nx_{n-3}+x_{n-1}x_{n-3}+1+a},\quad n=0,1,\ldots,\tag{E4}$$

where $a\in[0,+\infty)$ and the initial values $x_{-3},x_{-2},x_{-1},x_0\in R_+$.

In [8], Papaschinopoulos and Schinas investigated the global asymptotic stability of the following nonlinear difference equation:

$$x_{n+1}=\frac{\sum\limits_{i\in\mathbf{Z}_k-\{j-1,j\}}x_{n-i}+x_{n-j}x_{n-j+1}+1}{\sum\limits_{i\in\mathbf{Z}_k}x_{n-i}},\quad n=0,1,\ldots,\tag{E5}$$

where $k\in\{1,2,3,\ldots\}$, $\{j,j-1\}\subset\mathbf{Z}_k\equiv\{0,1,\ldots,k\}$ and the initial values $x_{-k},x_{-k+1},\ldots,x_0\in R_+$.

Recently, Li et al. [9] studied the global asymptotic stability of the following rational difference equation:

$$x_{n+1}=\frac{x_{n-k}^bx_{n-m}+x_{n-l}^b+a}{x_{n-k}^b+x_{n-m}x_{n-l}^b+a},\quad n=0,1,\ldots,\tag{E6}$$

where $a\ge 0$, $b\ge 0$, $k,l,m\in\{0,1,\ldots\}$ with $k\ne l$, $k\ne m$ and $m\ne l$, and the initial values are positive.

The main theorem in this note is motivated by the above studies. In this work, we consider the following nonlinear difference equation:

$$x_{n+1}=\frac{f(x_{n-r_1},\ldots,x_{n-r_k})g(x_{n-m_1},\ldots,x_{n-m_l})+h(x_{n-p_1},\ldots,x_{n-p_s})+1}{f(x_{n-r_1},\ldots,x_{n-r_k})+g(x_{n-m_1},\ldots,x_{n-m_l})+h(x_{n-p_1},\ldots,x_{n-p_s})},\quad n=0,1,\ldots,\tag{1}$$

where $f\in C(R_+^k,R_+)$, $g\in C(R_+^l,R_+)$ and $h\in C(R_+^s,[0,+\infty))$ with $k,l,s\in\{1,2,\ldots\}$, $0\le r_1<\cdots<r_k$, $0\le m_1<\cdots<m_l$ and $0\le p_1<\cdots<p_s$, and the initial values are positive real numbers.

2. Main result

In the sequel, write $u^*=\max\{u,1/u\}$ for any $u\in R_+$.

Lemma 1. (i) *Let* $u\ge w\ge 1$, $v\ge 1$ *and* $b\in[0,+\infty)$*; then* $(uv+1+b)/(u+v+b)\le(uv+1)/(u+v)$ *and* $(wv+1)/(w+v)\le(uv+1)/(u+v)$.

(ii) *Let* $u,v\in R_+$*; then* $(u/v)^*\le\max\{u^*/v^*,u^*v^*,v^*/u^*\}$.

Proof. (i) is obvious.

(ii) If $u/v\ge 1$, then it follows that

$$\left(\frac{u}{v}\right)^*=\frac{u}{v}=\begin{cases}\dfrac{u^*}{v^*}, & \text{if } u\ge v\ge 1,\\ u^*v^*, & \text{if } u\ge 1\ge v,\\ \dfrac{v^*}{u^*}, & \text{if } 1>u\ge v.\end{cases}$$

If $u/v<1$, then it follows that

$$\left(\frac{u}{v}\right)^*=\frac{v}{u}=\begin{cases}\dfrac{u^*}{v^*}, & \text{if } u<v\le 1,\\ u^*v^*, & \text{if } u\le 1\le v,\\ \dfrac{v^*}{u^*}, & \text{if } 1\le u<v.\end{cases}$$

Thus $(u/v)^*\le\max\{u^*/v^*,u^*v^*,v^*/u^*\}$. This completes the proof. □

Lemma 2. *Let $u, v \in R_+$ and $b \in [0, +\infty)$; then*

(i) $[(uv+1+b)/(u+v+b)]^* \le (u^*v^*+1)/(u^*+v^*) \le \min\{u^*, v^*\}$.
(ii) $[(uv+1)/(u+v)]^* = (u^*v^*+1)/(u^*+v^*)$.

Proof. We only prove (i) (the proof for (ii) is similar). Let $w = (uv+1+b)/(u+v+b)$; we have

$$w - 1 = \frac{(u-1)(v-1)}{u+v+b}. \tag{2}$$

If $(u-1)(v-1) \ge 0$, then by (2) we have $w \ge 1$, which implies

$$w^* = w = \frac{uv+1+b}{u+v+b} = \begin{cases} \dfrac{u^*v^*+1+b}{u^*+v^*+b}, & \text{if } u \ge 1 \text{ and } v \ge 1, \\ \dfrac{u^*v^*+1+u^*v^*b}{u^*+v^*+u^*v^*b}, & \text{if } u \le 1 \text{ and } v \le 1. \end{cases}$$

It follows from Lemma 1 that $w^* \le (u^*v^*+1)/(u^*+v^*)$.

If $(u-1)(v-1) < 0$, then by (2) we have $w < 1$, which implies

$$w^* = 1/w = \frac{u+v+b}{uv+1+b} = \begin{cases} \dfrac{u^*v^*+1+bv^*}{u^*+v^*+bv^*}, & \text{if } u > 1 \text{ and } v < 1, \\ \dfrac{u^*v^*+1+bu^*}{u^*+v^*+bu^*}, & \text{if } u < 1 \text{ and } v > 1. \end{cases}$$

Also it follows from Lemma 1 that $w^* \le (u^*v^*+1)/(u^*+v^*)$.

On the other hand, since $u^* \ge 1$ and $v^* \ge 1$, it follows that $(u^*v^*+1)/(u^*+v^*) \le u^*$ and $(u^*v^*+1)/(u^*+v^*) \le v^*$. This completes the proof. □

Now we formulate and prove the main result of this note.

Theorem 1. *Let f, g satisfy the following two conditions:*

(H_1) *There exists $F \in C(R_+^k, R_+)$ such that $[f(u_1, u_2, \ldots, u_k)]^* \le F(u_1^*, u_2^*, \ldots, u_k^*)$.*
(H_2) *$[g(u_1, u_2, \ldots, u_l)]^* = g(u_1^*, u_2^*, \ldots, u_l^*)$ and $g(u_1^*, u_2^*, \ldots, u_l^*) \le u_1^*$.*

Then $\overline{x} = 1$ is the unique positive equilibrium of Eq. (1) which is globally asymptotically stable.

Proof. Let $\{x_n\}_{n=-m}^{\infty}$ be a solution of Eq. (1) with initial conditions $x_{-m}, x_{-m+1}, \ldots, x_0 \in R_+$, where $m = \max\{r_k, m_l, p_s\}$. From (1), ($H_2$) and Lemma 2 it follows that for any $n \ge 0$,

$$\begin{aligned} 1 &\le x_{n+1}^* \\ &= \left[\frac{f(x_{n-r_1}, \ldots, x_{n-r_k})g(x_{n-m_1}, \ldots, x_{n-m_l}) + 1 + h(x_{n-p_1}, \ldots, x_{n-p_s})}{f(x_{n-r_1}, \ldots, x_{n-r_k}) + g(x_{n-m_1}, \ldots, x_{n-m_l}) + h(x_{n-p_1}, \ldots, x_{n-p_s})}\right]^* \\ &\le \frac{[f(x_{n-r_1}, \ldots, x_{n-r_k})]^*[g(x_{n-m_1}, \ldots, x_{n-m_l})]^* + 1}{[f(x_{n-r_1}, \ldots, x_{n-r_k})]^* + [g(x_{n-m_1}, \ldots, x_{n-m_l})]^*} \\ &\le [g(x_{n-m_1}, \ldots, x_{n-m_l})]^* \\ &= g(x_{n-m_1}^*, \ldots, x_{n-m_l}^*) \\ &\le x_{n-m_1}^*, \end{aligned} \tag{3}$$

from which we get that for any $n \ge 0$ and $0 \le i \le m_1$,

$$1 \le x_{i+(n+1)(m_1+1)}^* \le x_{i+n(m_1+1)}^*.$$

Let $\lim_{n\longrightarrow\infty} x_{i+n(m_1+1)}^* = A_i$ for any $0 \le i \le m_1$; then $A_i \ge 1 (0 \le i \le m_1)$. Write $M = \max\{A_0, A_1, \ldots, A_{m_1}\}$ and $A_{i+n(m_1+1)} = A_i$ for any integer $n (0 \le i \le m_1)$. Then there exists some $0 \le j \le m_1$ such that

$$\lim_{n\longrightarrow\infty} x_{j+n(m_1+1)}^* = M.$$

By (3) we have

$$x^*_{j+(n+1)(m_1+1)} \le g(x^*_{j+n(m_1+1)}, x^*_{j+(n+1)(m_1+1)-1-m_2}, \dots, x^*_{j+(n+1)(m_1+1)-1-m_l}) \\ \le x^*_{j+n(m_1+1)},$$

from which follows

$$M = g(M, A_{j-1-m_2}, \dots, A_{j-1-m_l}) = M.$$

Since

$$1 \le x^*_{n+1} \le \frac{[f(x_{n-r_1}, \dots, x_{n-r_k})]^*[g(x_{n-m_1}, \dots, x_{n-m_l})]^* + 1}{[f(x_{n-r_1}, \dots, x_{n-r_k})]^* + [g(x_{n-m_1}, \dots, x_{n-m_l})]^*},$$

from which with Lemma 1, (H_1) and (H_2) we get that

$$1 \le x^*_{n+1} \le \frac{F(x^*_{n-r_1}, \dots, x^*_{n-r_k})g(x^*_{n-m_1}, \dots, x^*_{n-m_l}) + 1}{F(x^*_{n-r_1}, \dots, x^*_{n-r_k}) + g(x^*_{n-m_1}, \dots, x^*_{n-m_l})}.$$

Therefore

$$M \le \frac{F(A_{j-1-r_1}, \dots, A_{j-1-r_k})g(M, A_{j-1-m_2}, \dots, A_{j-1-m_l}) + 1}{F(A_{j-1-r_1}, \dots, A_{j-1-r_k}) + g(M, A_{j-1-m_2}, \dots, A_{j-1-m_l})} \\ = \frac{F(A_{j-1-r_1}, \dots, A_{j-1-r_k})M + 1}{F(A_{j-1-r_1}, \dots, A_{j-1-r_k}) + M},$$

from which it follows that $M = 1$. This implies $A_i = 1$ for $0 \le i \le m_1$ and $\lim_{n\longrightarrow\infty} x^*_n = 1$. Since $1/x^*_n \le x_n \le x^*_n$, we obtain $\lim_{n\longrightarrow\infty} x_n = 1$. By (1) it follows that

$$1 = \frac{f(1, 1, \dots, 1)g(1, 1, \dots, 1) + h(1, 1, \dots, 1) + 1}{f(1, 1, \dots, 1) + g(1, 1, \dots, 1) + h(1, 1, \dots, 1)}.$$

Thus $\overline{x} = 1$ is the unique positive equilibrium of Eq. (1) and all of its solutions converge to 1. In the following we show that $\overline{x} = 1$ is locally stable.

For any $1 > \varepsilon > 0$, choose $\delta = \varepsilon/(1+\varepsilon)$ and let $\{x_n\}_{n=-m}^{\infty}$ be a solution of Eq. (1) with initial conditions $x_{-m}, x_{-m+1}, \dots, x_0 \in (1-\delta, 1+\delta)$. Then for any $-m \le i \le 0$, we have that $x_i < 1+\varepsilon$ and $1/x_i \le 1/(1-\delta) = 1+\varepsilon$. By (3) it follows that for any $n \ge 0$,

$$1 \le x^*_{n+1} \le x^*_{n-m_1} < 1+\varepsilon.$$

Thus we get that for any $n \ge 0$,

$$1-\varepsilon < \frac{1}{1+\varepsilon} \le \frac{1}{x^*_{n+1}} \le x_{n+1} \le x^*_{n+1} < 1+\varepsilon.$$

This implies that $\overline{x} = 1$ is globally asymptotically stable. This completes the proof. □

3. Examples

In this section, we shall give some applications of Theorem 1.

Example 1. Consider the equation

$$x_{n+1} = \frac{(x_{n-r_0}x_{n-r_1} + 1)g(x_{n-r_2}, \dots, x_{n-r_k}) + x_{n-r_0} + x_{n-r_1} + a}{x_{n-r_0}x_{n-r_1} + 1 + g(x_{n-r_2}, \dots, x_{n-r_k})(x_{n-r_0} + x_{n-r_1}) + a}, \quad n = 0, 1, \dots, \tag{4}$$

where $0 \le r_0 < r_1 < \dots < r_k$, $a \in [0, +\infty)$, the initial conditions $x_{-r_k}, \dots, x_0 \in R_+$ and $g \in C(R_+^{k-1}, R_+)$ satisfies $[g(u_2, u_3, \dots, u_k)]^* = g(u_2^*, u_3^*, \dots, u_k^*) \le u_2^*$. Then $\overline{x} = 1$ is the unique positive equilibrium of Eq. (4) which is globally asymptotically stable.

Proof. Let $F(x, y) = f(x, y) = (xy + 1)/(x + y)$ $(x > 0, y > 0)$ and $h(x, y) = a/(x + y)$ $(x > 0, y > 0)$. From Lemma 2, it follows that

$$[f(x, y)]^* = \frac{x^* y^* + 1}{x^* + y^*} = F(x^*, y^*).$$

Thus conditions (H_1) and (H_2) hold. By Theorem 1 we know that $\overline{x} = 1$ is the unique positive equilibrium of Eq. (4) which is globally asymptotically stable. □

Remark 1. Let $k = 2$ and $g(x) = x (x > 0)$; Eq. (4) reduces to Eq. (E3) and (E4).

Example 2. Consider the equation

$$x_{n+1} = \frac{x^b_{n-r_0} g(x_{n-r_2}, \ldots, x_{n-r_k}) + x^b_{n-r_1} + a}{x^b_{n-r_0} + g(x_{n-r_2}, \ldots, x_{n-r_k}) x^b_{n-r_1} + a}, \quad n = 0, 1, \ldots, \tag{5}$$

where $r_0, r_1, \ldots, r_k \in \{0, 1, \ldots\}$ with $r_i \neq r_j$ for $i \neq j$, $a, b \in [0, +\infty)$, the initial values are positive and $g \in C(R^{k-1}_+, R_+)$ satisfies $[g(u_2, u_3, \ldots, u_k)]^* = g(u^*_2, u^*_3, \ldots, u^*_k) \leq u^*_2$. Then $\overline{x} = 1$ is the unique positive equilibrium of Eq. (5) which is globally asymptotically stable.

Proof. Let $f(x, y) = (x/y)^b (x > 0, y > 0)$ and $h(y) = a/y^b (y > 0)$. From Lemma 1, it follows that

$$[f(x, y)]^* = \left[\left(\frac{x}{y}\right)^b\right] * \leq \max\left\{\frac{(x^*)^b}{(y^*)^b}, (x^* y^*)^b, \frac{(y^*)^b}{(x^*)^b}\right\} = F(x^*, y^*),$$

where $F(x, y) = \max\{\frac{x^b}{y^b}, (xy)^b, \frac{y^b}{x^b}\}$ $(x > 0, y > 0)$. Thus conditions (H_1) and (H_2) hold. By Theorem 1 we know that $\overline{x} = 1$ is the unique positive equilibrium of Eq. (5) which is globally asymptotically stable. □

Remark 2. Let $k = 2$ and $g(x) = x$ $(x > 0)$; Eq. (5) reduces to Eq. (E6).

Example 3. Consider the equation

$$x_{n+1} = \frac{x_{n-r_0} x_{n-r_1} + 1 + h(x_{n-r_2}, \ldots, x_{n-r_k})}{x_{n-r_0} + x_{n-r_1} + h(x_{n-r_2}, \ldots, x_{n-r_k})}, \quad n = 0, 1, \ldots, \tag{6}$$

where $r_0, r_1, \ldots, r_k \in \{0, 1, \ldots\}$ with $r_i \neq r_j$ for $i \neq j$, the initial values are positive and $h \in C(R^{k-1}_+, [0, +\infty))$. Then $\overline{x} = 1$ is the unique positive equilibrium of Eq. (6) which is globally asymptotically stable.

Proof. Let $F(x) = f(x) = g(x) = x (x > 0)$; it is obvious that conditions (H_1) and (H_2) hold. By Theorem 1 we know that $\overline{x} = 1$ is the unique positive equilibrium of Eq. (6) which is globally asymptotically stable. □

Remark 3. Let $h(u_2, \ldots, u_k) = u_2 + \cdots + u_k$ $(u_i > 0$ for $2 \leq i \leq k)$; Eq. (6) reduces to Eq. (E5).

References

[1] H.M. El-Owaidy, A.M. Ahmed, M.S. Mousa, On the recursive sequences $x_{n+1} = \frac{-\alpha x_{n-1}}{\beta \pm x_n}$, Appl. Math. Comput. 145 (2003) 747–753.
[2] C. Cinar, On the positive solutions of the difference equation $x_{n+1} = \frac{a x_{n-1}}{1 + b x_n x_{n-1}}$, Appl. Math. Comput. 156 (2004) 587–590.
[3] X. Li, Global asymptotic stability in a rational equation, J. Difference Equ. Appl. 9 (2003) 833–839.
[4] G. Ladas, Open problems and conjectures, J. Difference Equ. Appl. 4 (1998) 497–499.
[5] T. Nesemann, Positive nonlinear difference equations: Some results and applications, Nonlinear Anal. 47 (2001) 4707–4717.
[6] X. Li, Global behavior for a fourth-order rational difference equation, J. Math. Anal. Appl. 312 (2005) 555–563.
[7] X. Li, Qualitative properties for a fourth-order rational difference equation, J. Math. Anal. Appl. 311 (2005) 103–111.
[8] G. Papaschinopoulos, C.J. Schinas, Global asymptotic stability and oscillation of a family of difference equations, J. Math. Anal. Appl. 294 (2004) 614–620.
[9] Z. Li, D. Zhu, Global asymptotic stability of a higher order nonlinear difference equation, Appl. Math. Lett. 19 (2006) 926–930.